A more inclusive theory of numerals

1. Introduction – The aim is to provide a unified semantics for numerals which covers not only expressions of natural numbers, for example *two* (*2*), but also for those of decimal fractions, for example *two point five* (*2.5*). We claim that, e.g., sentences of the form *John read n Russian novels*, where *n* expresses a decimal fraction, are meaningful, in the sense that they support truth value judgments and logical inferences. Specifically, we make the following empirical claims.

Intuitions about truth conditions

(1) Scenario: John read 2 Pushkin novels, half of Anna Karenina, and half of War and Peace
- a. John read 2.5 Russian novels — TRUE
- b. John read 3 Russian novels — FALSE

(2) Scenario: John read 2 Pushkin novels and half of Anna Karenina, Mary read 2 Dostoyevsky novels and half of War and Peace
- a. John and Mary read 2.5 Russian novels each — TRUE
- b. John and Mary read 5 Russian novels together — FALSE

(3) Scenario: John read 2 Pushkin novels and half of Anna Karenina, Mary read 4 Dostoyevsky novels, half of Anna Karenina, and half of War and Peace
- a. John read 2.5 Russian novels and Mary read twice that amount — TRUE
- b. Mary read 5 Russian novels — FALSE

(4) Scenario: John read 2 Dostoyevsky novels and some of Anna Karenina, Mary read 2 Dostoyevsky novels and that part of Anna Karenina which John did not read
- a. If John read 2.75 Russian novels, then Mary read 2.25 Russian novels — TRUE
- b. If John read 2.75 Russian novels, then Mary read 2.5 Russian novels — FALSE

Intuitions about the validity of inferences

(5)
- a. John read 2.5 Russian novels $\Rightarrow$ John read 2.4 Russian novels — VALID
- b. John read 2.4 Russian novels $\not\Rightarrow$ John read 2.5 Russian novels — INVALID

While some of these judgments may be subtle, we note that there is a clear qualitative contrast between the sentences in (1)–(5) and the expression in (6), which is markedly unintelligible.

(6) *John read minus two Russian novels

Moreover, judgments about the sentences in (1)–(3) depend on how *Russian novel* is conceptualized in terms of how much structure Russian novels are conceptualized to have. That is, replacing *read n Russian novels* with e.g. *ate n apples* blurs the contrasts between the (a) and (b) sentences in (1)–(3) because *apple* is conceptualized as involving less structured objects than *Russian novel* (cf. the related contrast between *These salads contain apple* and #*These libraries contain Russian novel*).

The cardinality problem

The standard view on numerals shares with Frege (1884) the insight that numerical statements are, essentially, specifications of cardinalities of sets (cf. Montague 1973, Barwise and Cooper 1981, Heim and Kratzer 1998). The truth condition of *John read 2.5 Russian novels*, for example, would thus be either (7b) or (7c), depending on whether numerals are to have the 'at least' or the 'exact' reading as their lexical meaning.

(7)
- a. $|\{x : x \text{ is a Russian novel}\} \cap \{x : \text{John read } x\}| \geq 2.5$
- b. $|\{x : x \text{ is a Russian novel}\} \cap \{x : \text{John read } x\}| = 2.5$

Given that the cardinality of a (finite) set is a natural number, it follows that (7a) is equivalent to (8), and that (7b) is a contradiction (cf. Salmon 1997).

(8) $|\{x : x \text{ is a Russian novel}\} \cap \{x : \text{John read } x\}| \geq 3$

Thus, the cardinality analysis predicts, incorrectly, that *John read 2.5 Russian novels* is either equivalent to *John read 3 Russian novels*, or contradictory.

2. Proposal – Following Hackl (2000) and other works, we assume numerals denote degrees and merge with a null quantifier, MANY. We propose that MANY be defined as in (9) ($\sqcup$ and $\sqcap$ are the join and the meet operation, understood in the usual way, on $\langle \mathcal{D}_e \cup \{\emptyset\}, \sqsubseteq \rangle$, where $\sqsubseteq$ is the (individual) part-of relation and the empty set is by definition the least element of the partial order).

(9) $[\![\text{MANY}^c]\!](d)(A) = [\lambda x \in \mathcal{D}_e.\, \mu_A(x) \geq d]$

Thus, x is 'd-many A' iff x joined with an A atom is an A and $\mu_A(x)$ is at least d. The function $\mu_A(x)$ measures the "amount of A in x," so to speak, and is defined as follows.

(10) $$\mu_A(x) = \begin{cases} \mu_A(y) + 1 & \text{if } a \sqsubset x \text{ and } y \sqcup a = x \text{ for some } A \text{ atom } a \\ \mu_a(x) & \text{if } x \sqsubseteq a \text{ for some } A \text{ atom } a \end{cases}$$

Thus, $\mu_A(x)$ counts every A atom in x as 1 and every subpart of an A atom in x as $\mu_a(x)$, which is defined as in (11), where a is an arbitrary A atom.

(11) (i) μ_a is a function from $\{x \in \mathcal{D}_e \mid x \sqsubseteq a\}$ to $(0, 1] \cap \mathbb{Q}$
(ii) $\mu_a(x) + \mu_a(y) = \mu_a(x \sqcup y)$ for all $x, y \in \text{dom}(\mu_a)$ such that $x \sqcap y = \emptyset$
(iii) $\mu_a(a) = 1$

This means that (i) μ_a maps each part of an A atom a to the rational interval $(0, 1]$, (ii) (iii) μ_a is additive for any two discrete parts of a, and a is the unit of measurement.

3. Consequences – The definition of MANY in (9) in conjunction with the definition of μ in (10) and (11) makes it possible to count proper subparts of a novel by rational numbers between 0 and 1. It also entails that a proper subpart of a novel is not a novel, and neither is the combination of two halves of two different novels. In addition, it ensures that rational numbers between 0 and 1 are available for counting.

All observations in section 1 can be derived from (9). For instance, if p_1 and p_2 are two Pushkin novels, d_1 and d_2 two Dostoyevski novels, ak Anna Karenina, and wp War and Peace, then (2a) is true on the condition that John read $p_1 \sqcup p_2 \sqcup a'$ and Mary read $d_1 \sqcup d_2 \sqcup wp'$, where ak' is half of ak, i.e., where $\mu_{ak}(ak') = 0.5$ so that $\mu_{[\![\textit{Russian novel}]\!]}(p_1 \sqcup p_2 \sqcup ak') = 2.5$, and likewise for wp'. At the same time, (2b) is false since $\mu_{[\![\textit{Russian novel}]\!]}(p_1 \sqcup p_2 \sqcup d_1 \sqcup d_2 \sqcup ak' \sqcup wp') \neq 5$ since there is no atom $a \in [\![\textit{Russian novel}]\!]$ such that $\sqcup ak' \sqcup wp' \sqsubseteq a$. For lack of space, we will not present the other derivations here. We will note, however, that (9) does predict additional facts not presented in section 1. For example, it is predicted that for any predicate A whereby the concept of an atom makes no sense, e.g. whereby half an A is still an A, such expressions as *2.5 A* will be meaningless. The reason is that (9) entails that to be half is to be half of an atom. This prediction seems to be true, as evidenced by the oddness of (12a) and (12b). Note that *heap* and *amount* must be understood in their basic meaning, not in the coerced meaning as contextually specified quantities.

(12) a. #John has 2.5 heaps of sand
b. #John has 2.5 amounts of water

References

Barwise, John, and Robin Cooper. 1981. Generalized quantifiers and natural language. Linguistics and Philosophy 4:159–219.

Frege, Gottlob. 1884. Die Grundlagen der Arithmetik. Breslau: Verlage Wilhelm Koebner.

Hackl, Martin. 2000. Comparative quantifiers. Doctoral Dissertation, Massachusetts Institute of Technology.

Heim, Irene, and Angelika Kratzer. 1998. Semantics in Generative Grammar. Blackwell.

Montague, R. 1973. The proper treatment of quantification in ordinary English. Approaches to natural language 49:221–242.

Salmon, N. 1997. Wholes, Parts, and Numbers. Noûs Supplement: Philosophical Perspectives, 11, Mind, Causation, and World 31:1–15.