Beyond density

(former title: A plea for (no) monsters)

Andreas Haida
The Hebrew University of Jerusalem
andreas.haida@mail.huji.ac.il

Tue Trinh
University of Wisconsin at Milwaukee
tuetrinh@uwm.edu

MIT Workshop on Exhaustivity 2016
9/10/2016

Abstract Fox & Hackl (2006) proposes a theory of measurement which claims that **John has 3 children** is asymmetrically entailed by **John has 3.1 children**, or more generally, by any proposition of the form **John has n children** where **n** denotes a rational number greater than 3. In this talk, we present several observations on such sentences as **John has 3.1 children** and propose an analysis of numerals which accounts for them.

1 Introduction

1.1 Standard view on numerical statements

The standard view on numerical statements is based on Frege (1884), which takes these to be ascriptions of second order properties to concepts, or in more simplified terms, specifications of cardinalities of sets.

(1) John read 3 novels

1.1.1 The quantifier analysis

The numeral is treated as a determiner, of type $\langle\langle e,t\rangle,\langle\langle e,t\rangle,t\rangle\rangle$ (cf. Montague 1973; Barwise & Cooper 1981; Heim & Kratzer 1998).[1]

(2) a. $[\![\text{novels}]\!] = \{x : x \text{ is a novel}\} = \{a,b,c,\dots\}$
b. $[\![3]\!] = [\lambda P.\ \lambda Q.\ |P \cap Q| \geq 3]$
c. $[\![(1)]\!] = 1$ iff $|[\![\text{novels}]\!] \cap \{x : \text{John read } x\}| \geq 3$

1.1.2 The adjectival analysis

The numeral is treated as a modifier, of type $\langle\langle e,t\rangle,\langle e,t\rangle\rangle$ (cf. Link 1987; Landman 2003).[2]

(3) a. $[\![\text{novels}]\!] = \{x : x \text{ is a novel or a plurality of novels}\} = \{a,b,c,\dots,a\sqcup b, a\sqcup b\sqcup c,\dots\}$
b. $[\![3]\!] = [\lambda P.\ \lambda x.\ P(x) \wedge \#(x) = 3]$, where $\#(x) = |\{y : y \sqsubseteq x \wedge y \text{ is atomic}\}|$
c. $[\![(1)]\!] = 1$ iff $\exists x.\ x \in [\![\text{novels}]\!] \wedge \text{John read } x \wedge \#(x) = 3$

[1] We use "$\leq$" instead of "$=$" in (2-b) because we assume that the 'at least' meaning of numerals is basic, with the 'exact' meaning derived as a scalar implicature (cf. Horn 1972; Fintel & Heim 1997; Fintel & Fox 2002; Fox 2007). See Geurts (2006), Breheny (2008) for a different view.

[2] Henceforth, we use '$\sqcup$' to denote the join operation and '$\sqcap$' to denote the meet operation on $\langle \mathscr{D}_e \cup \{\emptyset\}, \sqsubseteq\rangle$, where '$\sqsubseteq$' is the (individual) part-of relation and the empty set is by definition the least element of the partial order. These operations are defined as usual: (i) $x \sqcup y = \iota\{z \mid x \sqsubseteq z \wedge y \sqsubseteq z \wedge \forall z'(x \sqsubseteq z' \wedge y \sqsubseteq z' \to z \sqsubseteq z')\}$, if defined; (ii) $x \sqcap y = \iota\{z \mid z \sqsubseteq x \wedge z \sqsubseteq y \wedge \forall z'(z' \sqsubseteq x \wedge z' \sqsubseteq y \to z' \sqsubseteq z)\}$, if defined.

1.1.3 The decimal problem

(4) John read 3.5 novels

By logical necessity, the cardinality of a set is a natural number.

(5) a. $|[\![\text{novels}]\!] \cap \{x : \text{John read } x\}| \geq 3.5 \Leftrightarrow |[\![\text{novels}]\!] \cap \{x : \text{John read } x\}| \geq 4$
b. $\exists x.\ x \in [\![\text{novels}]\!] \wedge \text{John read } x \wedge \#(x) = 3.5 \Leftrightarrow \bot$

Note that this problem cannot be solved by merely replacing '$=$' with '$\geq$' and vice versa.

(6) a. $|[\![\text{novels}]\!] \cap \{x : \text{John read } x\}| = 3.5 \Leftrightarrow \bot$
b. $\exists x.\ x \in [\![\text{novels}]\!] \wedge \text{John read } x \wedge \#(x) \geq 3.5 \Leftrightarrow \exists x.\ x \in [\![\text{novels}]\!] \wedge \text{John read } x \wedge \#(x) \geq 4$

1.2 Observations on decimals

1.2.1 Inferences

(7) a. John has 3.2 children $\Rightarrow$ John has 3.1 children
b. John has 3.1 children $\not\Rightarrow$ John has 3.2 children

1.2.2 Truth conditions

(8) John read 2 Pushkin novels, half of Anna Karenina, and half of War and Peace
a. John read 2.5 Russian novels ✓
b. John read 2.5 Russian novels and half of War and Peace ✓
c. John read 3 Russian novels ✗

(9) John read 2 Pushkin novels and half of Anna Karenina, Mary read 2 Dostoyevsky novels and half of War and Peace
a. John and Mary read 2.5 Russian novels each ✓
b. John and Mary read 5 Russian novels together ✗

(10) John read 2 Dostoyevsky novels and half of Anna Karenina, Mary read 4 Dostoyevsky novels, half of Anna Karenina, and half of War and Peace
a. John read 2.5 Russian novels and Mary read twice that quantity ✓
b. Mary did not read 5 Russian novels ✓

(11) John read 2 Dostoyevsky novels and some of Anna Karenina, Mary read 2 Dostoyevsky novels and that part of Anna Karenina which John did not read
a. If John read 2.75 Russian novels, then Mary read 2.25 Russian novels ✓
b. If John read 2.75 Russian novels, then Mary did not read 2.5 Russian novels ✓

1.2.3 Continuity

We take (12) to be a fact about natural language.

(12) Let r and r' be any two different rational numbers, $\phi[r]$ is meaningful iff $\phi[r']$ is

This means, more concretely, that replacing **2.5** in (13) with any other numeral which denotes a rational number will still result in a meaningful sentence, i.e. one which licenses inferences and whose truth condition can be specified.

(13) John read 2.5 Russian novels

1.3 Fox & Hackl (2006)

1.3.1 The Universal Density of Measurements

Fox & Hackl (2006) propose a theory which explains the following contrast, among others.

(14) a. John has 3 children $\rightsquigarrow$ ¬John has 4 children
b. John has more than 3 children $\not\rightsquigarrow$ ¬John has more than 4 children

Explanation: (14-a) can be consistently exhaustified while (14-b) cannot.

(15) a. exh(14-a) $\Leftrightarrow$ (14-a) $\wedge \neg\exists n \in \mathbb{Q}.\ n > 3 \wedge$ John has n children $\not\Leftrightarrow \bot$
b. exh(14-b) $\Leftrightarrow$ (14-b) $\wedge \neg\exists n \in \mathbb{Q}.\ n > 3 \wedge$ John has more than n children $\Leftrightarrow \bot$

Proof of (15-b): suppose (14-b) is true, then John has $3+\varepsilon$ children, which means he has more than $3+\varepsilon/2$ children, which means $\exists n \in \mathbb{Q}.\ n > 3 \wedge$ John has more than n children. □

Fox & Hackl claims to derive the fact that n ranges over $\mathbb{Q}$ and not $\mathbb{N}_0$ from the following postulate.

(16) The Universal Density of Measurements (UDM): measurement scales needed for natural language semantics are *always* dense.

"Without the UDM [...] [t]he set of degrees relevant for evaluation would be, as is standardly assumed, possible cardinalities of children (i.e., 1, 2, 3,...). The sentence would then assert that John doesn't have more than 4 children [...] If density is assumed, however, [...] the assertion would now not just exclude 4 as a degree exceeded by the number of John's children. It would also exclude any degree between 3 and 4" (Fox & Hackl 2006: 543).

1.3.2 Fox & Hackl's decimal problem

Under the assumption that exh operates on structurally defined alternatives (Katzir 2007; Fox & Katzir 2011), Fox & Hackl's theory presupposes that such sentences as (17) can be meaningful, since their negation can be.

(17) John has 3.5 children

However, it is not spelled out in Fox & Hackl (2006) what the truth condition for (17) is. In fact, these authors propose a way for (17) to be contextually equivalent to (18).

(18) John has 4 children

Specifically, they say that (19) is "a reasonable assumption about the granularity of measurement for collections of objects that are indivisible (based on world knowledge), such as children."

(19) Granularity for the measurement of collections of indivisible objects
xGy iff there is a natural number n, s.t. $x \in (n, n+1]$ and $y \in (n, n+1]$

1.3.3 Fox & Hackl's gap problem

Suppose the set of relevant degrees is (20).

(20) $S = \mathbb{Q} \backslash \{x \in \mathbb{Q} : 3 < x \leq 4\}$

This scale has a gap: missing from it are all the rational numbers greater than 3 and smaller than or equal to 4. Thus, 4 is missing from it, as well as 3.5, for example. Note, however, that the scale is dense: between any two points on it there is another point. Thus, the following sentence from Fox & Hackl (2006) is actually false: "If density is assumed [...] the assertion would [...] exclude any degree between 3 and 4" (Fox & Hackl 2006: 543).

Note that the scale in (20) still predicts (21) to have no implicature: (22) is still true.

(21) John read more than 3 novels

(22) exh(21) $\Leftrightarrow$ (21) $\wedge \neg\exists n \in S.\ n > 3 \wedge$ John read more than n novels $\Leftrightarrow \bot$

Proof of (22-b): suppose the scale is S and (21) is true, then John read $3+1+\varepsilon$ novels, which means he read more than $3+1+\varepsilon/2$ novels, and since $3+1+\varepsilon/2 \in S$, it follows that $\exists n \in S.\ n > 3 \wedge$ John read more than n novels. □

However, if S is the relevant scale, we predicts (23-a) to be meaningful but not (23-b).

(23) a. John read 4.5 novels
b. John read 3.5 novels

This conflicts with our claim about continuity in section 1.2.3.

2 Analysis

2.1 First attempt: A standard meaning

2.1.1 Definition

(24) $[\![\text{MANY}]\!](d)(A) = [\lambda x \in \mathscr{D}_e . A(x) \wedge \#_A(x) \geq d]$, where[3]

$$\#_A(x) = \begin{cases} \#_A(y)+1 & \text{if } a \sqsubset x \text{ and } y \sqcup a = x \text{ for some } A \text{ atom } a \\ 1 & \text{if } x = a \text{ for some } A \text{ atom } a \\ 0 & \text{otherwise} \end{cases}$$

(25) a is an A atom iff $A(a)$ is true and $\forall x \sqsubseteq a$, if $A(x)$ is true, then $x = a$

Informally, $[\![\text{MANY}]\!](d)(A)$ is true of an entity x iff the number of A atoms in x is equal to or greater than d. An A atom is an A with no proper subparts that are A.

2.1.2 Problem

It follows from (24) that $\#_A(x)$ is either 0, 1, or $n+1$, which means $\#_A(x)$ has to be a natural number. We then derive (26), where n ranges over $\mathbb{N}_0$.

(26) $\forall d_0, d_1 \in \mathbb{Q}$ such that $n < d_0, d_1 \leq n+1 : [\![\text{MANY}^c]\!](d_0)(A) = [\![\text{MANY}^c]\!](d_1)(A)$

Proof of (26): Assume that $[\![\text{MANY}^c]\!](d_i)(A)(x)$ is true for $i \in \{0,1\}$. Then $\#_A(x) \geq d_i$ and thus $\#_A(x) > n$, since $d_i > n$ by assumption. Since, furthermore, $\#_A(x) \in \mathbb{N}_0$, it follows that $\#_A(x) \geq n+1$. Hence, $\#_A(x) \geq d_j$ for $j = (i+1) \bmod 2$, since $n+1 \geq d_j$ by assumption. Consequently, $[\![\text{MANY}^c]\!](d_j)(A)(x)$ is true. □

We make the wrong prediction that (27) is true. This is because a natural number which is equal to or greater than 3.1 is just the same as a natural number which is equal to or greater than 3.2.

(27) John has 3.1 children $\Leftrightarrow$ John has 3.2 children

This would mean that both (28-a) and (28-b) are valid inferences, contrary to intuition.

(28) a. John has 3.2 children $\Rightarrow$ John has 3.1 children
b. John has 3.1 children $\Rightarrow$ John has 3.2 children

[3] Note that below and in the following discussion '0' and '1' always denote numbers and not truth values.

2.2 Second and final attempt: A non-standard meaning

2.2.1 Definition

(29) $[\![\text{MANY}^c]\!](d)(A) = [\lambda x \in \mathscr{D}_e. \exists a. a$ is an A atom $\wedge A(x \sqcup a) \wedge \mu_A(x) \geq d]$

a. $\mu_A(x) = \begin{cases} \mu_A(y)+1 & \text{if } a \sqsubset x \text{ and } y \sqcup a = x \text{ for some } A \text{ atom } a \\ \mu_a(x) & \text{if } x \sqsubseteq a \text{ for some } A \text{ atom } a \end{cases}$

b. For all A atoms a, μ_a is a function such that
   (i) $\mu_a : \{x \in \mathscr{D}_e \mid x \sqsubseteq a\} \to (0,1]$
   (ii) $\mu_a(a) = 1$, and
   (iii) $\mu_a(x) + \mu_a(y) = \mu_a(x \sqcup y)$ for all $x, y \in \text{dom}(\mu_a)$ such that $x \sqcap y = \emptyset$
   (iv) μ_a is a function onto $(0,1] \cap \mathbb{Q}$

A paraphrase of (29-b) is this: (i) μ_a maps each part of an A atom a to a member of $(0,1]$, (ii) a is the unit of measurement, (iii) μ_a is additive for any two discrete parts of a, and (iv) every rational number in $(0,1]$ is in its range.

2.2.2 Accounting for 1.2.1

(30) a. John has 3.2 children $\Rightarrow$ John has 3.1 children
b. John has 3.1 children $\not\Rightarrow$ John has 3.2 children

For every numeral n with denotation $d \in \mathbb{Q}$, we derive the following truth conditions:

(31) $[\![$John has n children$]\!] =$

$$= \exists x. \text{MANY}(d)([\![\text{children}]\!])(x) \wedge \text{John has } x$$
$$= \exists x \exists a. \text{At}_{[\![\text{children}]\!]}(a) \wedge [\![\text{children}]\!](x \sqcup a) \wedge \mu_{[\![\text{children}]\!]}(x) \geq d \wedge \text{John has } x$$

Accounting for (30-a)

- Trivially, $\forall d < d' : \mu_{[\![\text{children}]\!]}(x) \geq d' \Rightarrow \mu_{[\![\text{children}]\!]}(x) \geq d$
- If, furthermore, $n < d < d' \leq n+1$ for some $n \in \mathbb{N}$:

(32) $\exists x \exists a. \text{At}_{[\![\text{children}]\!]}(a) \wedge [\![\text{children}]\!](x \sqcup a) \wedge \mu_{[\![\text{children}]\!]}(x) \geq d' \wedge \text{John has } x \Rightarrow$
$\Rightarrow \exists x \exists a. \text{At}_{[\![\text{children}]\!]}(a) \wedge [\![\text{children}]\!](x \sqcup a) \wedge \mu_{[\![\text{children}]\!]}(x) \geq d \wedge \text{John has } x$

Accounting for (30-b)

Proof: Assume that $[\![\text{children}]\!] = \{x : x \text{ is a child or a plurality of children}\} = \{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}, \texttt{a} \sqcup \texttt{b}, \ldots\}$ and that for some part $\texttt{d}'$ of $\texttt{d}$ it holds that $\mu_{\texttt{d}}(\texttt{d}') = 0.1$.

- $[\![\text{children}]\!](\texttt{a} \sqcup \texttt{b} \sqcup \texttt{c} \sqcup \texttt{d}' \sqcup d) = [\![\text{children}]\!](\texttt{a} \sqcup \texttt{b} \sqcup \texttt{c} \sqcup \texttt{d}) = \text{truth}$
- $\mu_{[\![\text{children}]\!]}(\texttt{a} \sqcup \texttt{b} \sqcup \texttt{c} \sqcup \texttt{d}') = 3.1$

(33)
$$\begin{aligned} \mu_{[\![\text{children}]\!]}(\texttt{a} \sqcup \texttt{b} \sqcup \texttt{c} \sqcup \texttt{d}') &= 1 + \mu_{[\![\text{children}]\!]}(\texttt{b} \sqcup \texttt{c} \sqcup \texttt{d}') \\ &= 1 + 1 + \mu_{[\![\text{children}]\!]}(\texttt{c} \sqcup \texttt{d}') \\ &= 1 + 1 + 1 + \mu_{[\![\text{children}]\!]}(\texttt{d}') \\ &= 1 + 1 + 1 + \mu_{\texttt{d}}(\texttt{d}') \\ &= 1 + 1 + 1 + 0.1 \\ &= 3.1 \end{aligned}$$

Now assume that John has $\texttt{a} \sqcup \texttt{b} \sqcup \texttt{c} \sqcup \texttt{d}'$ and no other children or part of a child. Then, **John has 3.1 children** is true, since $\exists x \exists a. \text{At}_{[\![\text{children}]\!]}(a) \wedge [\![\text{children}]\!](x \sqcup a) \wedge \mu_{[\![\text{children}]\!]}(x) \geq 3.1 \wedge \text{John has } x$, viz. $\texttt{a} \sqcup \texttt{b} \sqcup \texttt{c} \sqcup \texttt{d}'$ for x and $\texttt{d}$ for a. At the same time, **John has 3.2 children** is false, since it is not the case that $\exists x. \mu_{[\![\text{children}]\!]}(x) \geq 3.2 \wedge \text{John has } x$.

2.2.3 Accounting for section 1.2.2

The first example

(34) John read 2 Pushkin novels, half of Anna Karenina, and half of War and Peace
a. John read 2.5 Russian novels ✓
b. John read 3 Russian novels ✗
c. John read 2.5 Russian novels and half of War and Peace ✓

(34-a) is true by virtue of either (i) and (ii):

i. 2 Pushkin novels ⊔ half of Anna Karenina ⊔ Anna Karenina are Russian novels
2 Pushkin novels ⊔ half of Anna Karenina is measured as 2.5 (given that half of Anna Karenina is measured as 0.5)
John read 2 Pushkin novels ⊔ half of Anna Karenina

ii. 2 Pushkin novels ⊔ half of War and Peace ⊔ War and Peace are Russian novels
2 Pushkin novels ⊔ half of War Peace is measured as 2.5 (given that half of War and Peace is measured as 0.5)
John read 2 Pushkin novels ⊔ half of War and Peace

(34-b) is false, since 2 Pushkin novels ⊔ half of Anna Karenina ⊔ half of War and Peace ⊔ any Russian novel are <u>not</u> Russian novels.

(34-c) is true, since both conjuncts are true. Importantly, the first conjunct is exhaustively true in the given situation if we add that John didn't read any other (part of a) Russian novel. Moreover, the prejacent of exhaustification is true by virtue of the facts in (i) alone. Thus, the truth of the second conjunct is logically independent from the exhaustive truth of the first.

In this connection, note the deviance of (35) (given the contextual information that Eugene Onegin is a Pushkin novel).

(35) #John read 2 Pushkin novels and Eugene Onegin

The deviance of (35) can be put down to the fact that if the first conjunct is exhaustively true then the second conjunct is either entailed to be true or entailed to be false. Hence, the second conjunct is either contradictory or redundant to the first.

The second example

(36) John read 2 Pushkin novels and half of Anna Karenina, Mary read 2 Dostoyevsky novels and half of War and Peace
a. John and Mary read 2.5 Russian novels each ✓
b. John and Mary read 5 Russian novels together ✗

(36-a) is true by virtue of the facts in (i) and (ii) (and the distribution provided by **each**):

i. 2 Pushkin novels ⊔ half of Anna Karenina ⊔ Anna Karenina are Russian novels
2 Pushkin novels ⊔ half of Anna Karenina is measured as 2.5
John read 2 Pushkin novels ⊔ half of Anna Karenina

ii. 2 Dostoyevsky novels ⊔ half of War and Peace ⊔ War and Peace are Russian novels
2 Dostoyevsky novels ⊔ half of War and Peace is measured as 2.5
Mary read 2 Dostoyevsky novels ⊔ half of War and Peace

Cumulatively, John and Mary did not read 5 Russian novels, since 2 Pushkin novels ⊔ 2 Dostoyevsky novels ⊔ half of Anna Karenina ⊔ half of War and Peace ⊔ any Russian novel are not Russian novels. Hence, (36-b) is false.

The third example

(37) John read 2 Dostoyevsky novels and half of Anna Karenina, Mary read 4 Dostoyevsky novels, half of Anna Karenina, and half of War and Peace
  a. John read 2.5 Russian novels and Mary read twice that quantity ✓
  b. Mary did not read 5 Russian novels ✓

The truth of the first conjunct of (37-a) can be derived from the following facts:

2 Dostoyevsky novels ⊔ half of Anna Karenina ⊔ Anna Karenina are Russian novels
2 Dostoyevsky novels ⊔ half of Anna Karenina is measured as 2.5
John read 2 Dostoyevsky novels ⊔ half of Anna Karenina

The second conjunct is true by virtue of the fact that Mary read twice that quantity:

i. Dostoyevsky novels `a` and `b` ⊔ half of Anna Karenina ⊔ Anna Karenina are Russian novels
Dostoyevsky novels `a` and `b` ⊔ half of Anna Karenina is measured as 2.5
Mary read 2 Dostoyevsky novels `a` and `b` ⊔ half of Anna Karenina
ii. Dostoyevsky novels `c` and `d` ⊔ half of War ans Peace ⊔ War and Peace are Russian novels
Dostoyevsky novels `c` and `d` ⊔ half of War and Peace is measured as 2.5
Mary read 2 Dostoyevsky novels `c` and `d` ⊔ half of War and Peace

The truth of (37-b) follows from the fact that 4 Dostoyevsky novels ⊔ half of Anna Karenina ⊔ half of War and Peace ⊔ any Russian novel are not Russian novels.

The fourth example

(38) John read 2 Dostoyevsky novels and some of Anna Karenina, Mary read 2 Dostoyevsky novels and that part of Anna Karenina which John did not read
  a. If John read 2.75 Russian novels, then Mary read 2.25 Russian novels ✓
  b. If John read 2.75 Russian novels, then Mary did not read 2.5 Russian novels ✓

If the part of Anna Karenina that John read is measured as 0.75, the antecedent of (38-a) and (38-b) is true by virtue of the following facts:

2 Dostoyevsky novels ⊔ the part of Anna Karenina that John read ⊔ Anna Karenina are Russian novels
2 Pushkin novels ⊔ the part of Anna Karenina that John read is measured as 2.75
John read 2 Pushkin novels ⊔ the part of Anna Karenina that he read

By the additivity of $\mu_{\text{Anna Karenina}}$, the part of Anna Karenina which John did not read is measured 0.25. Then, the truth of the consequent of (38-a) follows from the following facts:

2 Dostoyevsky novels ⊔ the part of Anna Karenina which John did not read ⊔ Anna Karenina are Russian novels
2 Pushkin novels ⊔ the part of Anna Karenina which John did not read is measured as 2.25
Mary read 2 Pushkin novels ⊔ the part of Anna Karenina which John did not read

The truth of the consequent of (38-b) follows from the same facts (partially derived from the additivity of $\mu_{\text{Anna Karenina}}$ if we add that Mary didn't read any other Russian novel or part of a Russian novel.

2.2.4 Accounting for 1.2.3

Continuity follows from stipulation (iv) in the definition of the function μ_a in (44).

(iv) μ_a is a function onto $(0,1] \cap \mathbb{Q}$.

The extension of $[\lambda d. \exists x. [\![\text{MANY}]\!](d)(A)(x) = 1]) = \mathbb{Q}$ if: For all $n \in \mathbb{N}_0 : \exists x \in A. \#_A(x) = n$, and (iv) is satisfied

2.2.5 Further predictions

"At least" and "exactly"

(39) John read 2 Pushkin novels, half of Anna Karenina, and half of War and Peace
  a. John read at least 2.5 Russian novels ✓
  b. John read exactly 2.5 Russian novels ✗

We can predict this contrast if we assume that the difference between **at least** and **exactly** amounts to the difference between ∃ and ∃!, i.e. if (39-a) and (39-b) have the truth conditions in (40-a) and (40-b), respectively.

(40) a. $\exists x.\ [\![\text{2.5 Russian novels}]\!](x) \wedge \text{John read } x$
  b. $\exists ! x.\ [\![\text{2.5 Russian novels}]\!](x) \wedge \text{John read } x$

We also predict (41) to be felicitous in the same scenario.

(41) There are two ways in which John read 2.5 Russian novels ✓

Wholeness

(42) a. John drank 0.5 bottles of beer
  b. #John drank 0.5 amounts of beer

We predict this contrast, due to the following results.

(43) a. $[\![\text{MANY}^c]\!](0.5)([\![\text{bottles of beer}]\!]) \neq \emptyset$
  b. $[\![\text{MANY}^c]\!](0.5)([\![\text{amounts of beer}]\!]) = \emptyset$

3 An existence proof

We have shown that the following definition of MANY can account for the observations that we made at the beginning.

(44) $[\![\text{MANY}]\!](d)(A) = [\lambda x \in \mathscr{D}_e . \mu_A(x) \geq d]$
  a. $\mu_A(x) = \begin{cases} \mu_A(y) + 1 & \text{if } a \sqsubset x \text{ and } y \sqcup a = x \text{ for some } A \text{ atom } a \\ \mu_a(x) & \text{if } x \sqsubseteq a \text{ for some } A \text{ atom } a \end{cases}$
  b. For all A atoms a, μ_a is a function such that
    (i) $\mu_a : \{x \in \mathscr{D}_e \mid x \sqsubseteq a\} \to (0,1]$
    (ii) $\mu_a(a) = 1$, and
    (iii) $\mu_a(x) + \mu_a(y) = \mu_a(x \sqcup y)$ for all $x, y \in \text{dom}(\mu_a)$ such that $x \sqcap y = \emptyset$
    (i) μ_a is a function onto $(0,1] \cap \mathbb{Q}$

This section is devoted to proving that the function μ_A exists, given certain metaphysical assumptions about the domain of individuals.

3.1 Infinite divisibility

The crucial metaphysical assumption that we need to make is that every entity is divisible into arbitrarily many discrete parts:[4, 5]

[4] It seems that a stricter condition might be desirable, viz. that every entity is arbitrarily divisible into discrete parts. However, such a condition would afford a notion of *possible division* of an entity and it is doubtful whether such a notion can be defined independently of the partial order $\langle \mathscr{D}_e \cup \{\emptyset\}, \sqsubseteq \rangle$.

[5] In order to simplify the formulation of the condition, we only assume that every entity is divisible into arbitrarily but *countably* many discrete parts.

(45) *Infinite Divisibility of Entities*
For all $x \in \mathscr{D}_e$ and $n > 1$, there is a set $S \subseteq \mathscr{D}_e$ such that $|S| = n$, $\bigsqcup S = x$, and $\bigsqcap S' = \emptyset$ for all $S' \subseteq S$ with $|S'| > 1$

According to (45), there are no smallest entities.

Furthermore, in conjunction with $\sqcup$-closedness (45) effects that there is no smallest difference between two parts of an entity.

Moreover, we need to specify the class of measurable predicates (where $\mathscr{P}_a = \{x \in \mathscr{D}_e \mid x \sqsubseteq a\}$):[6]

(46) A predicate $A \in \mathscr{D}_{et}$ is measurable iff (i) and (ii) hold:
(i) For all A atoms a and b, $a \sqcap b \neq \emptyset \Rightarrow a = b$
(ii) For all A atoms a the partial order $\langle \mathscr{P}_a, \sqsubseteq \rangle$ is a σ-algebra on $\langle \mathscr{D}_e \cup \{\emptyset\}, \sqsubseteq \rangle$

(47) A partial order $\langle A, \sqsubseteq \rangle$ is a σ-algebra on a lower bounded[7] partial order $\langle B, \sqsubseteq \rangle$ (with $A \subseteq B$) iff (i) it is upper bound, (ii) closed under complementation, and (iii) closed under countable joins
(i) $\langle A, \sqsubseteq \rangle$ is upper bounded iff $\bigsqcup A \in A$
(ii) $\langle A, \sqsubseteq \rangle$ is closed under complementation iff for all $x \in A$ there is a $y \in A$ such that $x \sqcup y = \bigsqcup A$ and $x \sqcap y = \bigsqcap B$
(iii) $\langle A, \sqsubseteq \rangle$ is closed under countable joins iff for all countable subsets S of A it holds that $\bigsqcup S \in A$

Thus, a predicate is measurable iff its atoms are pairwise discrete from each other and all atoms a have the following properties:

(i) The set of parts of a contains a greatest element (trivially satisfied, since a is a part of itself)
(ii) For every (proper) part of a, there is another part of a, discrete from the first, such that the two parts together are a
(iii) Countably many parts of a joined together are a part of a

With these metaphysical and linguistic assumptions in place, it follows that there are predicates A and measurement functions μ_A with $\text{ran}(\mu_A) = \mathbb{Q}$.

In the following proof sketch, we will make the following simplifying assumption:

(48) For all A atoms a, $\{x \in \mathscr{D}_e \mid x \sqsubseteq a\}$ is denumerable

We inductively define a sequence of partial functions from the set $\mathscr{P}_a$ of all parts of a onto $(0,1] \cap \mathbb{Q}$ (for each A atoms a):

- With a choice function, we select a maximal $\sqsubseteq$-path p from $\mathscr{P}_a$. In the induction base, p contains a as its first member.
  - By Cantor's theorem on countable dense orders, p can be mapped onto $(0,1] \cap \mathbb{Q}$, which gives us a measure for all members of p
  - The set of all complements of members of p form another infinitely descending path $\overline{p}$ which connects with a; their measure $1 - m$ is determined by the measure of a $(= 1)$ and their respective complement $(= m)$
  - The measure of the meet and the join of every pair of members of p and $\overline{p}$ is then determined by the already determined measures
- In the induction step, we show that if we successively remove maximal paths such as p and $\overline{p}$ from $\mathscr{P}_a$ then the measures in the remaining set are consistent with the the additivity requirement. Hence, by recursion we exhaust $\mathscr{P}_a$ and along the way define a total function from $\mathscr{P}_a$ onto $(0,1] \cap \mathbb{Q}$ that satisfies additivity.

[6] Alternatively, we could give a specification of the measurable part of any linguistic predicate. However, we suspect that the condition in (46) could be part of the characterization of count nouns.
[7] A partial order $\langle A, \sqsubseteq \rangle$ is lower bounded iff $\bigsqcap A \in A$

References

Barwise, John & Robin Cooper. 1981. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4. 159–219.

Breheny, Richard. 2008. A new look at the semantics and pragmatics of numerically quantified noun phrases. *Journal of Semantics* 25(2). 93–139.

Fintel, Kai von & Danny Fox. 2002. Pragmatics in Linguistic Theory. *MIT Classnotes* .

Fintel, Kai von & Irene Heim. 1997. Pragmatics in Linguistic Theory. MIT classnotes.

Fox, Danny. 2007. Pragmatics in Linguistic Theory. MIT classnotes.

Fox, Danny & Martin Hackl. 2006. The universal density of measurement. *Linguistics and Philosophy* 29. 537–586.

Fox, Danny & Roni Katzir. 2011. On the characterization of alternatives. *Natural Language Semantics* 19. 87–107.

Frege, Gottlob. 1884. *Die Grundlagen der Arithmetik*. Breslau: Verlage Wilhelm Koebner.

Geurts, Bart. 2006. Take 'five'. In Svetlana Vogeleer & Liliane Tasmowski (eds.), *Non-definiteness and plurality*, 311–329. Amsterdam: Benjamins.

Heim, Irene & Angelika Kratzer. 1998. *Semantics in Generative Grammar*. Blackwell.

Horn, Laurence. 1972. *On the semantic properties of the logical operators in english*: UCLA dissertation.

Katzir, Roni. 2007. Structurally-defined alternatives. *Linguistics and Philosophy* 30. 669–690.

Landman, Fred. 2003. Predicate-argument mismatches and the adjectival theory of indefinites. In Martine Coene & Yves D'hulst (eds.), *The Syntax and Semantics of Noun Phrases.* (Linguistic Today 55), Amsterdam and Philadelphia: John Benjamins.

Link, Godehard. 1987. Generalized quantifiers and plurals. In P Gärdenfors (ed.), *Generalized Quantifiers*, Dordrecht: Reidel.

Montague, R. 1973. The proper treatment of quantification in ordinary English. *Approaches to natural language* 49. 221–242.